

# Modular forms, modular symbols

(PARI-GP version 2.9.0)

## Modular Forms

To be completed later.

## Modular Symbols

Let  $G = \Gamma_0(N)$ ,  $V_k = \mathbf{Q}[X, Y]_{k-2}$ . We let  $\Delta = \text{Div}^0(\mathbf{P}^1(\mathbf{Q}))$ ; an element of  $\Delta$  is a *path* between cusps of  $X_0(N)$  via the identification  $[b] - [a] \rightarrow$  the path from  $a$  to  $b$ . A path is coded by the pair  $[a, b]$ , where  $a, b$  are rationals or  $\infty$ , denoting the point at infinity  $(1 : 0)$ .

Let  $\mathbf{M}_k(G) = \text{Hom}_G(\Delta, V_k) \simeq H_c^1(X_0(G), V_k)$ ; an element of  $\mathbf{M}_k(G)$  is a  $V_k$ -valued *modular symbol*. There is a natural decomposition  $\mathbf{M}_k(G) = \mathbf{M}_k(G)^+ \oplus \mathbf{M}_k(G)^-$  under the action of the  $*$  involution, induced by complex conjugation. The `msinit` function computes either  $\mathbf{M}_k$  ( $\varepsilon = 0$ ) or its  $\pm$ -parts ( $\varepsilon = \pm 1$ ) and fixes a minimal set of  $\mathbf{Z}[G]$ -generators ( $g_i$ ) of  $\Delta$ .

initialize $M = \mathbf{M}_k(\Gamma_0(N))^\varepsilon$	<code>msinit(<math>N, k, \{\varepsilon = 0\}</math>)</code>
the level $M$	<code>msgetlevel(<math>M</math>)</code>
the weight $k$	<code>msgetweight(<math>M</math>)</code>
the sign $\varepsilon$	<code>msgetsign(<math>M</math>)</code>
$\mathbf{Z}[G]$ -generators and relations for $\Delta$	<code>mspathgens(<math>M</math>)</code>
Decompose $p = [a, b]$ on the $(g_i)$	<code>mspathlog(<math>M, p</math>)</code>

### Create a symbol

Eisenstein symbol attached to cusp $c$	<code>msfromcusp(<math>M, c</math>)</code>
Cuspidal symbol attached to $E/\mathbf{Q}$	<code>msfromell(<math>E</math>)</code>
symbol having given Hecke eigenvalues	<code>msfromhecke(<math>M, v, \{H\}</math>)</code>
is $s$ a symbol ?	<code>msissymbol(<math>M, s</math>)</code>
the list of all $s(g_i)$	<code>mseval(<math>M, s</math>)</code>
evaluate symbol $s$ on path $p = [a, b]$	<code>mseval(<math>M, s, p</math>)</code>

### Operators

An operator is given by a matrix of a fixed  $\mathbf{Q}$ -basis.  $H$ , if given, is a stable  $\mathbf{Q}$ -subspace of  $\mathbf{M}_k(G)$ : operator is restricted to  $H$ .

matrix of Hecke operator $T_p$ or $U_p$	<code>mshecke(<math>M, p, \{H\}</math>)</code>
matrix of Atkin-Lehner $w_Q$	<code>msatkinlehner(<math>M, Q, \{H\}</math>)</code>
matrix of the $*$ involution	<code>msstar(<math>M, \{H\}</math>)</code>

### Subspaces

A subspace is given by a structure allowing quick projection and restriction of linear operators. Its fist component is a matrix with integer coefficients whose columns for a  $\mathbf{Q}$ -basis. If  $H$  is a Hecke-stable subspace of  $M_k(G)^+$  or  $M_k(G)^-$ , it can be split into a direct sum of Hecke-simple subspaces. To a simple subspace corresponds a single normalized newform  $\sum_n a_n q^n$ .

cuspidal subspace $S_k(G)^\varepsilon$	<code>mscuspidal(<math>M</math>)</code>
Eisenstein subspace $E_k(G)^\varepsilon$	<code>mseisenstein(<math>M</math>)</code>
new part of $S_k(G)^\varepsilon$	<code>msnew(<math>M</math>)</code>
split $H$ into simple subspaces (of $\dim \leq d$ )	<code>mssplit(<math>M, H, \{d\}</math>)</code>
$(a_1, \dots, a_B)$ for attached newform	<code>msqexpansion(<math>M, H, \{B\}</math>)</code>

### Overconvergent symbols and $p$ -adic $L$ functions

Let  $M$  be a full modular symbol space given by `msinit` and  $p$  be a prime. To a classical modular symbol  $\phi$  of level  $N$  ( $v_p(N) \leq 1$ ), which is an eigenvector for  $T_p$  with non-zero eigenvalue  $a_p$ , we can attach a  $p$ -adic  $L$ -function  $L_p$ . The function  $L_p$  is defined on continuous characters of  $\text{Gal}(\mathbf{Q}(\mu_{p^\infty})/\mathbf{Q})$ ; in GP we allow characters  $\langle \chi \rangle^{s_1} \tau^{s_2}$ , where  $(s_1, s_2)$  are integers,  $\tau$  is the Teichmüller character and  $\chi$  is the cyclotomic character.

The symbol  $\phi$  can be lifted to an *overconvergent* symbol  $\Phi$ , taking values in spaces of  $p$ -adic distributions (represented in GP by a list of moments modulo  $p^n$ ).

`mspadicinit` precomputes data used to lift symbols. If *flag* is given, it speeds up the computation by assuming that  $v_p(a_p) = 0$  if *flag* = 0 (fastest), and that  $v_p(a_p) \geq \textit{flag}$  otherwise (faster as *flag* increases).

`mspadicmoments` computes distributions *mu* attached to  $\Phi$  allowing to compute  $L_p$  to high accuracy.

initialize $Mp$ to lift symbols	<code>mspadicinit(<math>M, p, n, \{\textit{flag}\}</math>)</code>
lift symbol $\phi$	<code>mstooms(<math>Mp, \phi</math>)</code>
eval overconvergent symbol $\Phi$ on path $p$	<code>msomseval(<math>Mp, \Phi, p</math>)</code>
$mu$ for $p$ -adic $L$ -functions	<code>mspadicmoments(<math>Mp, S, \{D = 1\}</math>)</code>
$L_p^{(r)}(\chi^s)$ , $s = [s_1, s_2]$	<code>mspadicL(<math>mu, \{s = 0\}, \{r = 0\}</math>)</code>
$\hat{L}_p(\tau^i)(x)$	<code>mspadicseries(<math>mu, \{i = 0\}</math>)</code>

Based on an earlier version by Joseph H. Silverman  
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